A MODEL FOR IMPROVING PROSPECTIVE MATHEMATICS TEACHERS’ ALGEBRA CONTENT KNOWLEDGE

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This paper begins with a critical discussion of the theory/practice interface between theories of mathematics teacher education and what is actually achievable – given the constraints – in most mathematics teacher education programs. Then, data from 268 prospective middle-school mathematics teachers, who had completed full algebra programs in secondary schools in the United States of America, are summarised. Analyses reveal that most of these students, who were taking their last algebra course before they became fully qualified teachers of mathematics, did not know as much about linear and quadratic equations, and linear inequalities as might reasonably have been expected. Data, from a successful planned intervention, employing a so-called “5-R Intervention Model,” which aimed at upgrading the algebra content knowledge of the prospective teachers, are also analysed. The paper concludes with a plea for developing mathematics teacher education curriculum and teaching approaches that will enable mathematics educators to devise means by which prospective teachers of mathematics will not only become content-knowledgeable, but also become aware of when they know, and do not know, important areas of content.

Background: Identifying the Problem

Most people, at some stage in their lives, wish they knew more about something, or were able to do something better than they could in their present situation. In such circumstances, they may have partial knowledge, or partly developed skills, but these are not sufficient to meet the needs of the present situation. Such a state of affairs can arise, for example, when a person finds out that he or she needs a higher qualification to obtain a position that he or she desperately wants to have. Another common scenario arises in many countries where English is not the first language normally spoken, when young people who want to pursue higher studies in nations where higher education institutions require overseas students to reach a high level on an English language test, discover that they do not know English well enough to reach the level of competency being demanded of them. The question then arises: should I give up my quest to achieve my ambition, or should I take an intensive course aimed at helping me reach the required level of the “gatekeeper.”

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This paper is primarily about young people who wished to become middle-school teachers of mathematics. They had done well at school in their mathematical studies, and that was one of the reasons why they had decided to become specialist middle-school mathematics teachers. They were confident that, because they had always done well in mathematics at school, and because they really liked the subject, they would not have too much difficulty completing the necessary combination of mathematics, mathematics teacher education, and education studies that would enable them to become fully qualified, and competent middle-school teachers of mathematics. But, all of a sudden, in the middle of their teacher education studies, they found out that their knowledge of mathematics was not nearly as strong as they thought it was. Indeed, it was not strong enough for them to be able to teach mathematics competently. Understandably, they not only felt disappointed, but also confused and “let down by the system.” They knew that they needed to understand their mathematics at a much deeper level. They wondered why they had not discovered earlier that their mathematical knowledge was not as strong as they thought it was, and was certainly not as strong as it needed to be if they were to become effective teachers of middle-school mathematics.

This paper tells a story of 268 mostly young people, and two mathematics education instructors, at an American university with a strong reputation in mathematics teacher education. It tells how the students came to know that the level of their understanding of elementary algebraic equations and inequalities was much lower than they thought it was. It tells how they were initially confused when they learned this information, and how they followed an intervention program that, for almost all of them, brought their knowledge of elementary equations and inequalities to a level that would enable them to teach that topic competently – at least from a content perspective – to middle-school mathematics students.

It has been well established by research that, in many nations, university students who are preparing to become specialist teachers of mathematics often have unsatisfactory content knowledge with respect to the mathematics content that they will be expected to teach once they become full-time teachers (see, e.g., Clements, 2004; Suffolk, Tananone, & Clements, 2003). In the United States of America, the extent of the problem is acute. Schmidt et al. (2007), for example, reported a large quantitative study which showed that U.S. prospective middle-school teachers’ knowledge about “functions” – an important theme in middle and secondary school mathematics – was very low when compared with the knowledge of corresponding cohorts in Taiwan, South Korea, Bulgaria, Germany, and Mexico. Indeed, Scmidt et al. claimed, the US performance “lagged almost three fourths of a standard deviation below the international mean” (p. 1).

Over the past four years we (Clements and Ellerton) have been studying the algebra content knowledge and pedagogical content knowledge (PCK) of mathematics education students who were enrolled in an “Algebra for Teachers” second-year course (hereafter denoted AT2) at a large US university. We have been among those responsible for the mathematical preparation of cohorts of prospective middle-school teachers at this university. Faced with what we believed to be low standards of mathematics among the students whom we were studying (see, e.g., Vaiyavutjamai, Ellerton & Clements, 2005), we decided to attempt to explore the dimensions and extent of the problem, and to work towards improving the situation. We began gathering data that would inform us about what mathematics the prospective middle-school teachers had studied at school, and at college. We also sought data on their attitudes towards mathematics, towards teaching mathematics, and on how they viewed themselves mathematically. In particular, we were interested in the
levels of confidence they had when attempting questions that they might be expected to know when teaching middle-school algebra. An example of such a question would be to find all real-number values of \(x\) that would make \(x^2 = 9\) true. We expected that all of the students would state that \(x = 3\) was a solution, but were interested in what proportion would state that \(x = -3\) was also a solution. Furthermore, we wondered whether those students who stated that \(x = 3\) was a solution, but did not mention \(-3\) as a solution, had any idea that their answer was unsatisfactory (or, at best, only partly satisfactory).

In our quest to find out more, we designed a “background details” questionnaire and administered it to cohorts of students. This contained open-ended questions like “When you were at school, how did you feel about algebra?”, “Do you still feel that way about algebra?”, “What has been the greatest influence on your present view of algebra?”, “What aspect of school algebra did you understand best?”, “What aspect of school algebra did you have most trouble understanding?”, “At the present moment, what aspects of algebra do you like most?”, and “What aspects of algebra do you like best?” This questionnaire also asked students to list all the mathematics subjects they had studied at school and at university.

Written responses to this questionnaire made it clear that almost all of the students enjoyed and valued algebra, and had gained high grades in Algebra 1 and Algebra 2 at school. They all thought of themselves as being “good” at algebra, which is hardly surprising, since each of them intended to become a middle-school specialist teacher of mathematics.

Immediately after they had completed the questionnaire we asked the same students to solve 16 carefully selected equations and inequalities, and to respond to a quadratic scenario (the 16 questions, and the scenario are shown later in this paper – see Table 2 and Figure 2). Responses were scored, and out of a possible score of 20, the mean score, for the 268 prospective teachers, was 5.3. The 16 questions (and the quadratic scenario), were such that secondary school students who had completed Algebra 2 should have been able to solve all of them. Since all 268 prospective teachers had done well in Algebra 1 and 2 when in secondary school, one might have expected them not to have much difficulty with the questions. In addition to finding answers, the students were asked to indicate, for each question, how confident they were that they had given correct answers. In most cases they chose to respond: “I’m certain I’m right.” All of them had positive attitudes towards algebra, and were confident that they were able to obtain correct answers to the equations and inequalities and to the questions on the quadratic scenario.

Yet, deeper investigations that we carried out revealed that most of the 268 students had surprisingly poor algebra content knowledge. For example, many of the students who gave correct solutions to quadratic equations stated in the form \((x - a)(x - b) = 0\) thought that the \(x\) in \((x - a)\) and the \(x\) in \((x - b)\) were simultaneously equal to \(a\) and \(b\) respectively. We identified numerous fossilised algebraic misconceptions that guided the thinking of these prospective teachers. It seemed to us that teaching interventions that successfully corrected their faulty knowledge and skills, and their misconceptions was urgently needed (Clements & Ellerton, 2006; Vaiyavutjamai, Ellerton & Clements, 2005).

Clearly, we had identified a serious problem – most of these students did not know that they did not know. Even more seriously, we found ourselves faced with a crisis-of-confidence problem: once the students were informed of their scores out of 20 (on the equations and inequalities test, and the quadratic scenario), many of them felt shattered, and sometimes ashamed. They knew the questions that they had answered correctly were representative of algebra that, in about two years’ time, they might have to teach middle-
school students. And, now, apparently for the first time, they had discovered that they did not, in fact, know what that they thought they had known.

Our problem was to develop a method which would, simultaneously, restore their confidence and help them to know, definitely and accurately, the algebra they needed to know. There was another important dimension to the problem, however, and that related to the fact that the kind of algebra represented in the 16 questions and the quadratic scenario was not regarded as part of the algebra course that their mathematics instructors were expected to teach them. The attitude seemed to be that the students should already have learned that kind of algebra in secondary school. This meant that the instructors could only dedicate a relatively small amount of time to improving the students’ knowledge and skills with respect to equations and inequalities.

Towards a Solution to the Problem

Having identified the problem, our first step towards solving it was to consult relevant literatures related to algebra education and to mathematics teacher education. One possible response was to develop a problem-based learning (PBL) approach to the courses, in which equations and inequalities would arise in context, as students attempted to solve problems. We rejected that possibility because we believed that a problem-based curriculum would not allow us time to complete, comprehensively, the algebra curriculum that the State required of teacher education graduates. Furthermore, Kirscher, Sweller and Clark’s (2006) analyses raised doubts about the effectiveness of PBL unless students are strong in prerequisite knowledge, and have had experience in “prior structured experiences” (p. 82). Our data indicated that our students did not satisfy the conditions that Kirschner et al. laid down for a PBL approach to have a reasonable chance of being successful.

We examined Clarke and Hollingsworth’s (2002) interconnected model of Teacher Professional Growth, which took into account the “knowledge, beliefs and attitudes” of teachers (Clarke, 2009). However, since we had found that often the students with whom we were dealing “did not know that they did not know,” we felt we needed a theoretical approach which took greater account of that fact than did Clarke and Hollingsworth’s model. This led us to look carefully at Ponte and Chapman’s (2006) comments on the multitude of factors that can influence the effectiveness of mathematics teacher education programs. Ponte (2009) pointed to the influence of the students’ and teacher educators’ characteristics (including motives, interests, knowledge, beliefs and conceptions), the external constraints on a program, socio-cultural features of the society in which a program is offered, and career requirements and demands. Ponte called for programs that took account of the needs and interests of teacher education students, and for the effectiveness of teacher education programs to be evaluated. That seemed to fit our situation better than did Clarke and Hollingsworth’s model, but we felt that nevertheless it lacked the specific details needed to provide us with a sound teaching approach that took into account the interacting factors that we had identified.

The multi-dimensional model of mathematical knowledge for teaching put forward by Ball and her co-researchers (see, e.g., Hill, Ball & Schilling, 2004) distinguished between common content knowledge, knowledge of the mathematical horizon, specialised content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum (Ball, Thames, Bass, Sleep, Lewis, & Phelps, 2009). Although these subdivisions distinguished between different types of knowledge relevant to
prospective (and practising) teachers, they seemed not to take any account of the immediacy, or urgency, of the situation that we had uncovered with the prospective middle-school teachers. Indeed, none the established models or theories that we consulted addressed the fragile balance between the prospective teachers’ knowledge of algebra and their concerns, even embarrassment, over the possibility that they would not be ready to teach because they had recently discovered that they did not know content that they were supposed to know.

The 5-R Intervention Model

**Ideal Background Circumstances in Which the Model can be Applied**

In the study described in this paper the instructors needed to respond to an education situation in which their students – who were all prospective teachers of mathematics – were, at least initially, not aware of the fact that they did not have an adequate knowledge of certain elementary algebra knowledge, skills and concepts relating to linear and quadratic equations and inequalities. These prospective teachers had studied both linear and quadratic equations and inequalities in Algebra 1 and Algebra 2 classes when they were in secondary school, and therefore did know something about them. However, they had forgotten important details, and what remained was often inaccurate, vague and disconnected. The students’ prior engagement in learning of similar knowledge, skills, and concepts, needed to be taken into account in the planning of any program aimed at raising the students to the level of mastery and understanding that might be expected of them, in about two years’ time, when they would graduate and become qualified middle-school teachers of algebra.

The attempt to help the students learn satisfactorily the elementary algebra skills relating to equations and inequalities also needed to take account of the small amount of curriculum time available. In fact, linear and quadratic equations and inequalities were not really included in the documented intended curriculum. Yet, both instructors recognised that any attempt merely to “cram” elementary algebraic knowledge and skills “into the students’ heads” would be less than useful, for the students needed to understand the material and retain that understanding when they graduated and became full-time teachers.

Another feature called for in the design of the intervention program was that it was important that students quickly rid themselves of any embarrassment and shame that they might have felt as a result of their realisation that they did not really know the algebra that they thought had known.

In summary, to have a chance of being effective, an intervention strategy needed to help students not only to learn what they needed to learn quickly and effectively (i.e., with understanding), but also to convince them, after their new set of learning experiences, that they really did know what they now believed they knew. It was very important that they came to believe, with justification, that from a content perspective, at least, they were capable of teaching linear and quadratic equations and inequalities.

**Components of the Model**

The “5-R Intervention Model” which we developed in response to the set of circumstances we have just described, is illustrated in Figure 1. This Model features five ordered treatment components (which will be referred to as the 5-R components) – Realise, Review, Reflect, Revisit and Retain. Each component requires action on a student’s part.
The Need to Focus

The 5-R Intervention Model is not meant to be suitable in all mathematics teaching/learning contexts. It is meant to be pertinent when a teacher wishes to focus on some particularly important area of mathematics content knowledge that, his or her pedagogical content knowledge suggests, students are not likely to understand, even though they need to understand it. It is particularly useful in circumstances where students have partial knowledge, and partly formed concepts, but feel an imperative to upgrade their knowledge so that it incorporates a deeper level of understanding.

In the study described in this paper the content focus area was “elementary linear and quadratic equations and inequalities.” The students felt an imperative to understand this important theme because they knew that as future middle-school teachers of algebra they needed to have a strong understanding of equations and inequalities. Once they were made aware that their knowledge, skills and concepts in the focus area were shaky, they wanted to improve them. It should be noted, here, that the 5-R Intervention Model has also been applied, successfully, for other content focus areas – for example, it has been used with prospective middle-school teachers who needed to deepen their understanding of links between the various ways of representing linear and quadratic functions.

Transfer of Ownership

When designing the intervention program we regarded it as important that somehow we would be able to transfer the intellectual ownership of the intervention process from the instructors to the teacher education students themselves. From many perspectives, not least their own, it was desirable that the students improve their knowledge, skills and concepts, and overall understanding of the topic.
We believed that if fossilised misconceptions were guiding a student’s thinking then the only way for this state of affairs to be permanently changed was if the student himself or herself made an effort to identify those misconceptions, and then took steps to eradicate them from his or her concept images. In order for that to happen, a student needed to know why he or she had made errors, and what the appropriate procedures might be, and why these procedures were appropriate. Doing the pencil-and-paper tests, and then receiving their scripts back, fully corrected, at the next session was an important component of this identification and correction phase of the intervention strategy, as was the whole-class review sessions that took place immediately after the corrected scripts were returned to the students. Obviously, the written reflections which followed were also important. Students were informed that five percent of the total grade for the subject would be based on the quality of their reflections.

Components of the 5-R Intervention Model – Realise, Review, Reflect, Revisit, and Retain
The five components of the intervention can be summarised in the following way:

1. There is an initial reality check, by which students come to Realise that their thinking on the area under consideration has been guided by fossilised misconceptions. This often generates some confusion for students who are confronted with the reality that they do not know what they thought they knew.

2. There is a Review component, by which student misconceptions are identified and corrected by an instructor, with students being guided toward appropriate conceptions.

3. Students then Reflect by making written statements about how they previously had thought about the concepts, and about the corrected, revised and extended conceptual understandings that they are in the process of developing.

4. Then follows a component when students Revisit over an extended period – usually between 6 and 12 weeks – their conceptual understandings. The intention, here, is to ensure that new understandings are appropriate, stable and coherent.

5. The final component is when students are assessed to see if they have acquired and Retained accurate conceptions, and whether they can apply, with appropriate confidence, their new understandings in relevant problem-solving or problem-posing contexts.

As a consequence of each treatment component, students are depicted (see Figure 1) as reaching new conceptual phases (or “C-phases”) in their development of relational understandings that are internalised in the form of accurate, rich, linked concept images. The conceptual phases are termed “Certainty (Misplaced),” “Confusion,” “Construction,” “Conceptualisation” “Coherence,” and “Confidence.”

We now consider more fully the R-components of the model.

The Realisation component. Many AT2 students began with misplaced confidence regarding the extent of their understanding of linear and quadratic equations and inequalities, which had been chosen as the content focus. As one student wrote:

This was an interesting exercise because I thought I knew the answers, when I did not. Also, I began to understand algebra in a way that I can remember it more clearly and actually understand it, instead of just using rules to find the answer.
After the students responded to content questions on a carefully developed pencil-and-paper test, all of the students experienced a reality check in the sense that they obtained wrong answers to questions which were certain they knew how to do correctly.

In their reflections, many of the students noted that previously they had rarely encountered equations like $x + 5 = 8 - (3 - x)$, and that therefore they had not proceeded appropriately when they arrived at a statement like “$x + 5 = 5 + x$.” Similarly, when trying to solve $4(x + 1) = 4(x - 3)$, they said they had never before been asked to interpret a proposition like “$4 = -12$.” However, after their thinking about equations such as these had been straightened out, in the whole-class reviews, they became confident that they would be able to deal with such problems in the future. On the retention test they demonstrated that this confidence was well placed.

Although they had been told that their scores on this initial test would not “count” towards their final AT2 grade, many of the students became worried at this stage, because they now realised that they did not really understand what they thought they had understood. Their state of induced anxiety was heightened by their recognition of the fact that, yes, if and when they did become middle-school algebra teachers, they would be expected to have a strong understanding of elementary equations and inequalities.

The Review component. Once the students’ test scripts had been returned to them, fully assessed, a brief review period (spread over one 2-hour class) immediately followed. During this session, the instructor attempted to help the students to recognise fossilised misconceptions and to develop appropriate concepts. In the first reflection, written during the week after the review session, many students wrote comments similar to the following:

Completing the problems on equations and inequalities on the first day of class made me take a step back and think. I had not taken an algebra course since my junior year of high school, and what I needed to do for each problem was fuzzy. On some I could remember that there was a rule that went with them, but figuring it out and remembering it was difficult. When we got our papers back, fully corrected, I was shocked to see I only got 6 out of 20; I thought I had done much better than that. As we went through the answers some information came back to me but mostly I saw how to do these problems in a whole new way. For the first time I actually understood the reasoning and wasn’t just applying some rule because I was told to do so.

I wrote this reflection after doing the homework set of problems just to see if I could say that I can now do these problems. I am pleased to say that, yes, I now know how to do them, and understand what I am doing. Doing that homework made me realise that it’s not just a question of following through a sequence of steps. I needed to do that, but I also needed to work through and personalise the reasoning! For the first time I actually feel I have learned to understand what is really going on with algebra problems involving square roots and inequalities! I feel much better about answering questions on them and explaining them. I find it very sad that it took until college for me to begin to understand these concepts and I don’t want to do that to my students. I am so happy a teacher actually gave me the opportunity to learn and not just memorise!

Comments in the third column of Table 1 suggest the kind of teaching which occurred in the whole-class review sessions. When commonly-held misconceptions were revealed by student responses at the pre-teaching stage, these were dealt with in the whole-class review
sessions. The emphasis in these sessions was on helping students to think *holistically* about the given equations and inequalities. That is to say, there was an emphasis on relational understanding (Skemp, 1976). It was also emphasised that, often, there is more than one good way of solving a problem. Thus for example, the third task in Table 1 might be satisfactorily approached by dividing both sides of the inequality by 9, to get the equivalent inequality $x + 1 > x - 2$, which is obviously true for all real-number values of $x$.

### Table 1
Answers to and Comments on Three Elementary Algebra Tasks

<table>
<thead>
<tr>
<th>Equation/ Inequality</th>
<th>Correct Answer</th>
<th>Examples of Comments Made (by the Instructors) in the Whole-Class Sessions Held Immediately after the Students’ Test Scripts had been Returned</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + 3 &gt; 6 - (3 - x)$</td>
<td>“This is not true for any real-number value of $x$.”</td>
<td>“Since $6 - (3 - x)$ equals $6 - 3 + x$, which equals $3 + x$, the given inequality is equivalent to $x + 3 &gt; 3 + x$. But, this is not true for any real-number value of $x$, because $x + 3$ is always equal to $3 + x$ (by the commutative law for addition). Hence the original inequality has no real-number solutions.”</td>
</tr>
<tr>
<td>$4(x+1) = 4(x - 3)$</td>
<td>“No real-number value of $x$.”</td>
<td>“Since $4(x+1)$ and $4(x - 3)$ equal $4x + 4$, and $4x - 12$, respectively, the original inequality is equivalent to $4x + 4 = 4x - 12$. This has no real-number solution because adding 4 cannot have the same effect as subtracting 12. Hence the original equation has no real-number solution.”</td>
</tr>
<tr>
<td>$9(x+1) &gt; 9(x - 2)$</td>
<td>“All real-number values of $x$ makes this true.”</td>
<td>“Since $9(x+1)$ and $9(x - 2)$ equal $9x + 9$, and $9x - 18$, respectively, for any real-number value of $x$, it follows that $9(x+1)$ must be greater than $9(x - 2)$ for all real-number values of $x$. Adding 9 to $9x$ must always give a larger value than subtracting 18 from $9x$.”</td>
</tr>
</tbody>
</table>

### The Reflection component.

After the 2-hour review, in class, was completed, each student was required to write and submit a reflective overview of his or her performance on the original test. In these reflections students were expected to discuss each question that they had answered incorrectly. They were expected to state the answers they had given, and the correct answer. With questions they had answered incorrectly, they were expected to provide evidence that they had engaged in forms of metacognition by reflecting on why they had not chosen solution pathways that might have led them to correct answers. They were expected to identify, in their original solution attempts, the use of incorrect knowledge, or inappropriate or inaccurate skills, or incorrect interpretations.

In their reflections, students often commented specifically on the power of the reflection process. One student wrote, for example: “This reflection helped me out a lot because it made me make sense of the algebra, now.” Both the review teaching session and the reflection component became vital, and complementary, aspects of the students’ re-learning process (Artzt & Armour-Thomas, 1997). This was summed up well by the student who wrote:
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This test really showed me that I do not understand mathematics as well as I thought I did. All along I have just been going through the motions and solving problems without even thinking about what is really happening in the problem. I did not know why I was doing certain things. I just knew that that’s what I was supposed to do. Now I have a better understanding of equations and inequalities and why they are solved the way they are. I think that this knowledge and understanding will be a great foundation to everything else that I will do, and teach, mathematically.

The students were required to submit their reflections to their instructor. They were told that each reflection would be graded, and would contribute up to 5 percent of the final grade for the course. It was made clear to the students that they were expected to provide evidence that they had engaged in forms of self analysis (the word “metacognition” was not used) by reflecting on why they had chosen to do the tasks in the ways shown on their test scripts and, in cases for which incorrect answers were obtained, why they had not chosen methods that might have led them to correct answers. They were expected to identify, in their original solution attempts, the use of incorrect knowledge, or inappropriate or inaccurate skills, or incorrect interpretations. The students were also asked to comment on questions that they had answered correctly but had done so without having full understanding.

The Revisiting component. During their reflections many students acknowledged that they needed “more practice problems” in the area of equations and inequalities. During the 14 weeks of the semester that remained after the reflection stage, the AT2 instructors often revisited concepts involving equations and inequalities – usually, but not always, in the contexts of topics being studied at the time. Immediately after the reflections, students were asked to do some additional homework tasks on equations and inequalities, and this provided an opportunity for the students to demonstrate to themselves that they had developed significantly in their understandings. Later in the semester, when linear and quadratic functions were being studied, there were numerous opportunities for students to apply their new-found understandings of linear equations and inequalities. The same was true when students studied linear and geometric sequences. All AT2 students had to work in pairs on a problem-solving and problem-posing project, and most of the projects required them, at some point, to solve equations and/or inequalities – and, as a result, the students were able to apply their found-found knowledge, skills and concepts in problem situations.

Thus, the students were able to revisit the knowledge, skills and concepts they had constructed, and this helped them to consolidate what they had learned about equations and inequalities, and to develop richer, more comprehensive and accurate concept images.

The Retention component. After they had completed their first reflection the teachers and the students moved on to the stipulated intended curriculum for AT2. The students were not told that their knowledge and concepts relating to linear equations and inequalities would be formally tested, once more, towards the end of the semester. But that is what happened, for at the end of the AT2 course, all students were asked to answer a set of questions parallel to those they had originally answered at the beginning of the semester. Since the students were not warned that this retention test was coming, it was a genuine test of retention. No student ever complained about having to do the test, for they saw it as providing an opportunity for them to demonstrate that they had improved. Subsequent analyses showed that most students improved their understandings of equations and inequalities over the course of the
semester. In fact, none of the 268 students gained a lower score (out of a possible 20) on the retention test than on the initial test.

The students appreciated that this retention assessment enabled them to be reasonably certain that their recently developed understandings were appropriate, and that they could and should be confident that they now had the content knowledge and conceptual understandings needed to teach the subject effectively. At the end of the 16-week semester, a student wrote, in a separate reflection exercise:

Taking AT2 has been one of the greatest experiences that I have had in math throughout my college life. I have learned many different concepts and ideas in this class that I never knew or understood before.

Another student was moved to reflect, directly, on what had happened to her in relation to equations and inequalities:

I believe that I’ve grown in my understanding of algebra because of this class. One thing that stands out to me is the pre-test we took during the first class. Once we got the test back graded and went over the answers I realised that I made a lot of stupid silly mistakes. I didn’t recognise that \( x + 1 > x - 2 \) is always true and other similar problems. Now, I actually look at the problem and analyse it to see if what the problem is asking makes sense.

Another student had cause to reflect on her school algebra experiences:

During middle school and high school, algebra was one of my favorite subjects. I liked being able to follow rules and to easily find a solution to any problem. However, I now know the reason I was good at algebra was because I could memorise and remember rules. I had very little understanding to back up these rules and I had no idea why they worked. This course has helped me to build this understanding and to explain why the rules worked. It forced me to solve the problem on my own without relying on the formula. This way of thinking gives meaning to the problem rather than just mindlessly plugging numbers into a formula in order to get an answer.

For much of the remainder of this paper data arising from the four-year application of the 5-R model to a total of 10 AT2 classes (with 268 students, altogether), will be presented and analysed. In every one of the 10 classes, middle-school mathematics teacher-education students were taking their last algebra course before they would become full-time elementary or middle-school teachers of mathematics.

Almost all of the students wrote, in their reflections, that when they first saw the 16 equations and inequalities on the tests they were not too concerned because they seemed to be “basic” or “easy.” Many said that even when they were answering the questions they thought that they were giving correct responses. Too often, however, their thinking with respect to many of the questions was inappropriate. For example, one student wrote:

Once I got the results I was shocked at how low my score was. I had completely forgotten the basic principles. These basic rules date back to middle-school years. It took me a while to appreciate that, yes, my answers were not correct. I will not make these mistakes again. So far as Carrie’s attempt to solve the quadratic equation was concerned [see Figure 2], I simply did not recognise Carrie’s mistakes. Although I now can see that she did not need to distribute the terms in the parentheses at the start, I’m still not completely sure what she did wrong when she wrote \( 0 \times 0 = 0 \) in her check.
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The final comment in this last statement emphasises that sometimes the instructors had to talk with students individually to help overcome misconceptions. The student who wrote the last excerpt was struggling to recognise why the statement “0 × 0 = 0” in the check indicated a serious lack of understanding. It was only after one of the instructors talked individually with the student that the misconception was straightened out. Thus, the 5-R Intervention Model should not be interpreted as providing a rigid, almost depersonalised, approach to remediation. It was true, nevertheless, that in most cases it was not necessary for an instructor to talk with individual students about their misconceptions.

The Study Design

The main purpose of the study was to assist prospective elementary- and middle-school teachers to become aware of their misconceptions on basic equations and inequalities, and then to help them correct those misconceptions. The study included a retention aspect of checking whether the correct conceptions were retained over a period of between 6 and 12 weeks. An important aspect of the intervention program was the creation of an environment in which all students would reflect metacognitively on the strategies they used when they attempted to solve equations and inequalities.

Population and Sample Considerations

Altogether, 268 students, in 10 “Algebra for Teachers” (AT2) classes, participated in the study during the period 2006-2009. The great majority of these students were prospective middle-school mathematics teachers, and their involvement in the study was a direct result of their enrolment in the second-year, semester-long, AT2 course offered within a mathematics department at a large North American university. Between 2006 and 2009 most of the AT2 classes at that university were taught by the same two instructors (Instructor A and Instructor B), both of whom were well qualified and experienced mathematics educators. For five of the six semesters there were only two sections for AT2, one of which was taught by Instructor A and the other by Instructor B. In each section there were between 25 and 30 students. For each of the 10 classes, the mean pre-intervention score was between 4.8 and 7.1 (out of 20). The highest pre-intervention score gained by a student, out of 20, was 16.

AT2 was a compulsory subject for students seeking an endorsement to teach middle-school mathematics, and students involved in the study had committed themselves to a teaching career in which the primary content focus would be middle-school mathematics. In written responses to items on a questionnaire, administered at the beginning of each semester, almost all of the students stated that they liked algebra, and had been among the top algebra students in the secondary schools that they had attended.

For each semester, placement of students into the two AT2 classes was done by departmental administrators, without consultation with the instructors. Initial testing revealed that for each semester when there two AF2 sections, the students in each section had very similar mathematics profiles. On initial tests given to both sections mean test scores were never very different. Although both authors followed the same “approved” AT2 curriculum, the order in which topics were taught was not the same.

From a “population” statistical perspective, there is no reason to suppose that the students taking AT2 could be regarded as representative of any well-defined group, except
perhaps of students at the particular university who were preparing to become middle-school mathematics specialists. The university at which the study occurred has a very strong reputation in mathematics education, and it is likely that most of the 268 students were better qualified, and mathematically more advanced, than “typical” students preparing to be middle-school mathematics teachers at most other universities. Because of the difficulty of defining a population for the study, and because the students were not randomly assigned to the classes, it would be difficult to defend, statistically, the use of inferential statistical procedures. Hence, such procedures are not used in this paper.

Table 2
The Eight Equations and Eight Corresponding Inequalities

<table>
<thead>
<tr>
<th>(Pre-Teaching)</th>
<th>(Pre-Teaching)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{x} = 3 )</td>
<td>( x &gt; 4 )</td>
</tr>
<tr>
<td>( x^2 = 9 )</td>
<td>( x^2 &gt; 4 )</td>
</tr>
<tr>
<td>( x = \frac{9}{x} )</td>
<td>( x &gt; \frac{4}{x} )</td>
</tr>
<tr>
<td>( x^2 + 6 = 0 )</td>
<td>( x^2 + 2 &gt; 0 )</td>
</tr>
<tr>
<td>( 9(x - 1) = 0 )</td>
<td>( 4(x - 1) &gt; 0 )</td>
</tr>
<tr>
<td>( 4(x + 1) = 4(x - 3) )</td>
<td>( 9(x + 1) &gt; 9(x - 2) )</td>
</tr>
<tr>
<td>( (x - 3)(x - 2) = 0 )</td>
<td>( (x - 3)(x - 1) &gt; 0 )</td>
</tr>
<tr>
<td>( x + 5 = 8 - (3 - x) )</td>
<td>( x + 3 &gt; 6 - (3 - x) )</td>
</tr>
</tbody>
</table>

The Main Pencil-and-Paper Instruments

In the second of 32 AT2 sessions, each student attempted to solve eight equations and eight inequalities (see Table 2). They were asked to state all real number values of \( x \) which would make statements true. The students were also asked to indicate, for each task, how confident they were that the solution they obtained was correct. For each task they had to select one of five possible indicators: “I’m certain I’m correct,” “I think I’m correct,” “I’ve got a 50-50 chance of being correct,” “I think I’m wrong,” and “I’m certain I’m wrong.”

These 16 tasks were carefully constructed so that for each equation there was a corresponding inequality. Thus, for example, \( x^2 + 6 = 0 \) and \( x^2 + 2 > 0 \) were regarded as forming a pair. Many of the tasks invited students to reflect on meaning: thus, for example, whereas \( x^2 + 6 = 0 \) has no real-number solutions, \( x^2 + 2 > 0 \) has infinitely many. In a similar vein, \( x + 5 = 8 - (3 - x) \) is an identity with infinitely many solutions, but the corresponding inequality, \( x + 3 > 6 - (3 - x) \), has no real-number solution.

After the students had completed the 16 tasks they were asked to respond, in writing, to the quadratic scenario shown in Figure 2. Each response to the quadratic scenario (in Figure 2) was given a score from 0 to 4, depending on how many of the following four points were noted by the student:

- Lines 2, 3, and 4 were unnecessary.
- In Lines 5 through 7, the word “or”, and not “and,” should have been used.
- For the check, the solutions should have been substituted into Line 1.
Applying the 5-R intervention model

- For the check, each solution should have been substituted into both parentheses in the initial equation.

Students were asked to solve \((x + 2)(2x + 5) = 0\), and then to check their answer. One student, Carrie, wrote the following (line numbers have been added):

\[
\begin{align*}
(x + 2)(2x + 5) &= 0 & \text{Line 1} \\
\therefore 2x^2 + 5x + 4x + 10 &= 0 & \text{Line 2} \\
\therefore 2x^2 + 9x + 10 &= 0 & \text{Line 3} \\
\therefore (2x + 5)(x + 2) &= 0 & \text{Line 4} \\
\therefore (2x + 5) &= 0 \text{ and } (x + 2) &= 0 & \text{Line 5} \\
\therefore 2x &= -5 \text{ and } x &= -2 & \text{Line 6} \\
\therefore x &= -\frac{5}{2} \text{ and } x &= -2 & \text{Line 7}
\end{align*}
\]

Check: Put \(x = -\frac{5}{2}\) in \((2x + 5)\), and put \(x = -2\) in \((x + 2)\).

Thus, when \(-\frac{5}{2}\) and \(x = -2\), \((2x + 5)(x + 2)\) is equal to \(0 \times 0\) which is equal to 0.

Since 0 is on the right-hand side of the original equation, it follows that \(x = -\frac{5}{2}\) and \(x = -2\) are the correct solutions.

Comment fully on Carrie’s responses.

**Figure 2.** The quadratic scenario.

Between 6 and 12 weeks after students had been administered the initial pencil-and-paper instruments the same students were administered, without warning, parallel forms of the original instruments. The first and third columns of Table 3 show the original and parallel equations and inequalities. In the parallel form of the quadratic scenario, a hypothetical student made the same four kinds of incorrect responses when attempting to solve the equation \((2x - 3)(x + 3) = 0\).

Throughout most of the 14 weeks the normal curriculum for the algebra course was followed. The focus was on using algebra in problem-solving contexts, and not on tasks directly resembling the pre-teaching pencil-and-paper tasks.

**Quantitative Analyses**

Entries in Tables 3 and 4 indicate that between the administrations of the pre-teaching equations and inequalities tasks and the initial quadratic scenario, and the parallel retention tasks, most students learned how to cope well with the algebra, irrespective of which teacher (Instructor A or Instructor B) was teaching their AT2 class.

Before the study began the authors hypothesised that at the pre-teaching stage most of the mathematics teacher education students would not recognise some or all of the incorrect statements in the quadratic scenario. Entries in the fourth column in Table 4 show that this hypothesis was confirmed. At the retention stage, however, a healthier situation prevailed – irrespective of which instructor had taught the students (see column 5 in Table 4). Entries in Table 4 strongly suggest that the effects of teaching on performance – on solving equations and inequalities and on responses to the quadratic scenario – for the AT2 students taught by the two instructors were similar. Students with both instructors tended to show large, and approximately equal, improvements on equations and inequalities and on the quadratic scenario.
Table 3
Percentages Correct, of 268 Mathematics Teacher Education Students on 16 Pre-Teaching and 16 Parallel Retention Tasks (8 Equations, 8 Inequalities at Each Stage; n = 268)

<table>
<thead>
<tr>
<th>Pre-Teaching Question</th>
<th>Pre-Teaching, Number &amp; % Correct</th>
<th>Post-Teaching Retention Question</th>
<th>Retention, Number &amp; % Correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \frac{1}{x} = 3 )</td>
<td>209 (78%)</td>
<td>( \frac{1}{x} = 2 )</td>
<td>257 (96%)</td>
</tr>
<tr>
<td>2. ( x^2 = 9 )</td>
<td>59 (22%)</td>
<td>( x^2 = 16 )</td>
<td>255 (95%)</td>
</tr>
<tr>
<td>3. ( x = \frac{9}{x} )</td>
<td>48 (18%)</td>
<td>( x = \frac{4}{x} )</td>
<td>236 (88%)</td>
</tr>
<tr>
<td>4. ( x^2 + 6 = 0 )</td>
<td>59 (22%)</td>
<td>( x^2 + 3 = 0 )</td>
<td>252 (94%)</td>
</tr>
<tr>
<td>5. ( 9(x-1) = 0 )</td>
<td>257 (96%)</td>
<td>( 5(1-x) = 0 )</td>
<td>258 (96%)</td>
</tr>
<tr>
<td>6. ( 4(x+1) = 4(x-3) )</td>
<td>138 (51%)</td>
<td>( 3(x-1) = 3(x+3) )</td>
<td>265 (99%)</td>
</tr>
<tr>
<td>7. ( (x-3)(x-2) = 0 )</td>
<td>161 (60%)</td>
<td>((x+3)(x+4) = 0)</td>
<td>256 (96%)</td>
</tr>
<tr>
<td>8. ( x + 5 = 8 - (3 - x) )</td>
<td>58 (22%)</td>
<td>( 2x + 5 = 10 - (5 - 2x) )</td>
<td>254 (95%)</td>
</tr>
<tr>
<td>9. ( \frac{1}{x} &gt; 4 )</td>
<td>4 (1%)</td>
<td>( \frac{1}{x} &gt; 3 )</td>
<td>144 (54%)</td>
</tr>
<tr>
<td>10. ( x^2 &gt; 4 )</td>
<td>13 (5%)</td>
<td>( x^2 &gt; 9 )</td>
<td>206 (77%)</td>
</tr>
<tr>
<td>11. ( x &gt; \frac{4}{x} )</td>
<td>1 (0%)</td>
<td>( x &gt; \frac{1}{x} )</td>
<td>113 (42%)</td>
</tr>
<tr>
<td>12. ( x^2 + 2 &gt; 0 )</td>
<td>40 (15%)</td>
<td>( x^2 + 5 &gt; 0 )</td>
<td>242 (90%)</td>
</tr>
<tr>
<td>13. ( 4(x-1) &gt; 0 )</td>
<td>167 (62%)</td>
<td>( 4(x-3) &gt; 0 )</td>
<td>253 (94%)</td>
</tr>
<tr>
<td>14. ( 9(x+1) &gt; 9(x-2) )</td>
<td>63 (24%)</td>
<td>( 9(x+5) &gt; 9(x+7) )</td>
<td>266 (99%)</td>
</tr>
<tr>
<td>15. ( (x-3)(x-1) &gt; 0 )</td>
<td>1 (0%)</td>
<td>((x+3)(x-2) &gt; 0)</td>
<td>161 (60%)</td>
</tr>
<tr>
<td>16. ( x + 3 &gt; 6 - (3 - x) )</td>
<td>85 (32%)</td>
<td>( x + 2 &gt; 7 - (5 - x) )</td>
<td>260 (97%)</td>
</tr>
</tbody>
</table>

Table 4
Pre-Teaching and Retention Class data for the 268 Students on the Equation and Inequality Tasks, and on the Quadratic Scenario for Classes Taught by the Two Teachers

<table>
<thead>
<tr>
<th>Groups Taught by Teacher A and Teacher B</th>
<th>Pre-Teaching Mean Score and SD on 16 Questions (/16)</th>
<th>Retention Mean Score and SD on 16 Parallel Questions (/16)</th>
<th>Pre-Teaching Mean Score and SD on the Quadratic Scenario (/4)</th>
<th>Retention Mean Score and SD on the Parallel Quadratic Scenario (/4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher A (141 Students)</td>
<td>4.18 (2.27)</td>
<td>14.58 (2.55)</td>
<td>1.07 (0.63)</td>
<td>3.27 (0.86)</td>
</tr>
<tr>
<td>Teacher B (127 Students)</td>
<td>4.29 (2.16)</td>
<td>14.63 (2.67)</td>
<td>1.10 (0.67)</td>
<td>3.28 (0.80)</td>
</tr>
<tr>
<td>Overall (268 Students)</td>
<td>4.23 (2.59)</td>
<td>14.60 (2.85)</td>
<td>1.08 (0.74)</td>
<td>3.27 (0.95)</td>
</tr>
</tbody>
</table>

A sceptical reader might notice that in the absence of control groups there could be no guarantee that t students, who did not participate in a 5-R intervention program but
completed a “normal” AT2 curriculum, would not have shown the same levels of improvement as those indicated in Tables 3 and 4. The main evidence against such an interpretation can be found in the strength of the convictions expressed by students in their reflections about the value of the initial test followed by the review session, followed by their personal, written reflections.

Some relevant quantitative evidence on the issue was obtained when one of the two instructors found himself teaching a Number Theory class with 34 students, of whom 15 had studied AT2 with either Instructor A or Instructor B as the teacher, and the other 19 students had studied AT2 under a different instructor (neither Instructor A nor Instructor B) who had used a problem-based approach to curriculum design and instruction. All 34 students agreed to take a parallel form of the “equations and inequalities” test. It was found that the mean score for the 15 students who had, earlier, followed the 5-R program was 14.1 out of a possible 16 (standard deviation, 1.8), and the mean for the remaining 19 students was 5.7 (standard deviation, 2.1). Yet, these two samples of students performed equally well on all other aspects of the Number Theory course.

Although these two samples of 19 and 15 students within the Number Theory class were not the result of random assignment, it might be useful to regard the two samples as representative of two populations of prospective middle-school teachers. The sample with 19 students could be thought of as representing a population comprising prospective teachers who had not had the opportunity to study equations and inequalities according to the 5-R Model, and the sample with 15 students could be thought of as representing a population comprising prospective teachers who had had the opportunity to study equations and inequalities according to the 5-R Model. If the null hypothesis was that there would be no difference in mean performance of the populations, and the distribution of differences between the means of the independent samples was as per Student’s $t$, with the number of degrees of freedom equal to 32 ($= 19 + 15 - 2$), then, with homogeneity of variance assumed, a pooled variance $t$-value of between 12 and 13 would be obtained (Popham & Sirotnik, 1973). This could be associated with a statistically significance difference in the means ($p < 0.0001$, 2-tailed).

The 19 students in the Number Theory course who had not taken the 5-R program performed only slightly better than the 268 students in the 5-R classes on the pre-intervention test questions (that is to say, before the review and before the reflection). This suggests that students completing an AT2 course that did not include a 5-R component would have begun their teaching careers not understanding fundamental aspects of equations and inequalities.

**Student Confidence**

On the pre-teaching test, for each equation, all 268 students indicated that either they were certain they were right or they thought they were right. They were less confident on the inequalities, but usually thought they were right. For all eight inequality tasks, 245 of the 268 students indicated that they were certain they were right, or they thought they were right, or they thought they had a 50-50 chance of being correct. This initial confidence was usually misplaced, suggesting that many students “did not know that they did not know.”

For the retention test, most students indicated that they were confident they were correct on 14 of the 16 tasks. This time their confidence was vindicated. The two exceptions were for $1/x > 3$, and $x > 1/x$. On the post-teaching/Retention tests, 62 of the 268 students indicated that they were certain their answer was correct on these two tasks. In fact, 39 of
those 62 students did give correct responses to those tasks. Most of the other 206 students were not confident that their answers were correct for those two tasks, and in most cases these students gave incorrect responses. At least, these students now “knew that they did not know”.

**Other 5-R Intervention Studies with Younger Students: A Summary**

Space does not permit us to report on other intervention studies that we have conducted using the 5-R Intervention Model. Two of these studies involved 9th-grade and 11th-grade students, mainly Hispanic or African American, attending schools in low socio-economic areas of Chicago. Three other studies involved students from schools in low socio-economic areas who were about to begin university studies, and our task was to provide intensive courses in number sense and elementary algebra so that they would be better placed with the quantitative and algebraic demands that would be placed on them in their university studies. With each group, initial testing revealed an extremely low mean performance on the pre-teaching test, but after whole-class reviews, and student reflections on “why and where I went wrong,” the student groups made large mean gains on parallel post-teaching tests.

We have found that the greater imperative students feel to do well in the subject, because of some well-defined need, the greater effort they will put into their written reflections, and the greater will be the extent of their improvement. One important aspect of the 5-R Intervention strategy is that it can easily fit into typical school or college programs without taking much time from the normal curriculum. The keys to success seem to be the identification of a well-defined, unambiguous focus, and a well-balanced and well-structured combination of teaching, testing, and student reflection. No single component of the intervention strategy is more important than another. Use of the strategy can motivate students to learn content well. It can also give them confidence and, because it emphasises understanding rather than recall, result in their retaining what they learn.

**Helping Prospective Teachers to Find the Right Track**

We have found that giving prospective middle-school teachers of mathematics permission to reflect on their learning of mathematics often results in their making profound statements about their own education. Someone who has not used the 5-R Intervention Model might think that the Model represents a rigid, instrumental approach to issues surrounding the identification and remediation of low-level mathematics weaknesses. Our experience has been that it provides students with a vehicle through which they can identify and correct their own fossilised misconceptions, develop appropriate concepts, and become aware when they did not fully understand mathematics concepts.

It will be appropriate to close this paper with some words written by a prospective teacher who scored a total of 8 out of a possible 20 on the 16 equations and inequalities and the quadratic scenario. In his written reflection this student felt moved to look back on his own mathematics education, and to contemplate what might happen in schools if ... He wrote:

When I was working on these basic algebra questions, two words can be used to describe how I was feeling ... brain lapse! I knew that I had learned how to do every one of these problems at some point in my educational career. However, I could not remember the processes. Looking back, I realise that I never truly learned how to do
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these problems at all. I simply memorised rules that led me blindly to the answers. I never gained the understanding of how the problems worked. This is a big problem in our educational system. It concentrates more on the memorisation or rules and facts rather than on getting at the reasoning behind the problems. ...

For me, mathematics has always been a rather enjoyable subject. When I know how to solve the problems and my numbers work, I get satisfaction out of achieving the correct answers. Every school year I would “build” on what I already “knew” about mathematics. Ironically, what I supposedly already knew had to be reviewed at the start of the year before moving on. After a quick refresher on the material, I quickly remembered [original emphasis] how to do it and I moved through the twenty-some practice problems for homework. What is the problem with this? We had to spend time on the material that we had supposedly already “learned” the year before! Really, we were only drilled with the steps to solve multiple mathematics problems. If those steps didn’t stick, we had to be reminded again. Unfortunately, our education system doesn’t spend enough (or hardly any) time on teaching the reasoning of mathematics to students. If educators spent a little more time on this, more than likely the time spent at the beginning of each school year on review could be spent on actual new material. It seemed to me that if we followed this plan, we may accomplish a whole lot more!

In his reflections this student made clear that he was now confident that he understood the mathematics behind 14 of the 16 equations and inequalities, and the quadratic scenario. However, he admitted that although he now knew how to get correct answers for the two most difficult of the inequalities (Question 9, \( \frac{1}{x} > 4 \) and Question 11, \( x > \frac{4}{x} \)), he “did not understand fully the reasoning of it all,” and still felt that his “ability to do these types of problems is purely based on knowing the ‘rules’. ” This student had learned to “know when he didn’t know,” as well as “to know when he knew.”

The initial responses to the 16 equations and inequalities generated much evidence that these prospective teachers, who were taking their last algebra course before they became full-time teachers of middle-school mathematics, often had no idea that they were making errors. Thus, for example, at the pre-teaching stage only 22 percent of the students stated that \( x = -2 \) was a solution to \( x^2 = 4 \), and all the others indicated that they were “certain that they right” when they stated that \( x = 2 \) was the only real-number solution to this equation. On the retention test, only 13 of the 268 students did not state both solutions to \( x^2 = 16 \), and all 13 indicated that they were not sure that the single solution (\( x = 4 \)) that they offered represented the correct answer. Content-wise, these 13 students fell short of being ready to teach middle-school algebra – would you want your Grade 8 child to be taught algebra by someone who didn’t immediately recognise that \( x^2 = 16 \) has two real-number solutions? – but at least, by realising that \( x = 4 \) may not be the only solution they had progressed beyond the state of ignorance when they did not know that they did not know.

We have found that the 5-R Intervention Model has the power to help students pass from darkness into light, and has placed many prospective teachers of middle-school mathematics on a new track that will lead them into being knowledgeable and competent teachers.
References


